THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2058 Honours Mathematical Analysis I
Tutorial 5

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1. Let (x_n) be a bounded sequence of real numbers and let $s \in \mathbb{R}$. Show that $\overline{\lim} x_n \leq s$ if and only if for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $x_n < s + \varepsilon$ for all $n \geq N$.

2. Show the following using both the closed and bounded definition of a compact set and the open cover definition of a compact set:

   (a) if A is non-empty and compact, then $\sup A$ exists and $\sup A \in A$;

   (b) if A is compact and if $B \subset A$ is closed, then B is compact. *Hint: the complement of a closed set is open.*

3. (Cantor's Intersection Theorem) Prove the following generalization of the Nested Interval Theorem for compact sets: Suppose $\{K_n\}_{n=1}^{\infty}$ is a sequence of nested non-empty compact subsets of $\mathbb{R}$. Then $\bigcap\limits_{n=1}^{\infty} K_n \neq \emptyset$.

1. Let (x_n) be a bounded sequence of real numbers and let $s \in \mathbb{R}$. Show that $\overline{\lim} x_n \leq s$ if and only if for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $x_n < s + \varepsilon$ for all $n \geq N$.

Pf: $\limsup_n x_n = \inf_n \sup_{k \geq n} x_k$.

$\Rightarrow$: First sps $\limsup x_n \leq s$. Then by we have

$$\inf_n \left\{ \sup_{k \geq n} \{x_k\} \right\} \leq s$$

So by property of inf, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, s.t. $\sup_{k \geq N} x_k < s + \varepsilon$.

Since sup is an u.b., this means $x_k < s + \varepsilon$ for all $k \in \mathbb{N}$.

$\Leftarrow$: Suppose for given $\varepsilon > 0$, $\exists N$ s.t. $x_k < s + \varepsilon$ for all $k \geq N$.

So $s + \varepsilon$ is an u.b. of the set $\{x_k : k \geq N\}$. By sup, have

$$\sup_{k \geq N} x_k < s + \varepsilon.$$

So $\sup_{k \geq N} x_k$ is l.b. of the set $\{s + \varepsilon : \varepsilon > 0\}$.

So taking inf on both sides gives

$$\inf_N \sup_{k \geq N} x_k \leq s.$$

2. Show the following using both the closed and bounded definition of a compact set and the open cover definition of a compact set:

   (a) if A is non-empty and compact, then $\sup A$ exists and $\sup A \in A$;

   (b) if A is compact and if $B \subset A$ is closed, then B is compact. *Hint: the complement of a closed set is open.*

Pf: a). Closed and bounded: A cpt, so it is bdd. so $\sup A$ exists. WTS $\sup A \in A$.

We'll show that $\sup A$ is a limit pt. of A

Let $\varepsilon > 0$. Then $\exists a_\varepsilon \in A$, s.t.

$$\sup A - \varepsilon < a_\varepsilon < \sup A + \varepsilon.$$

$$\Updownarrow$$

$$|\sup A - a_\varepsilon| < \varepsilon.$$

i.e. $\sup A$ is a limit pt. of A. Then since A is closed, $\sup A \in A$.

Open Cover: A is cpt iff. if $\{U_i\}_{i \in I}$ is an open cover of A, $\exists i_1, \ldots, i_N$ s.t. $U_{i_1}, \ldots, U_{i_N}$ cover A.

First need to show that A is bdd. Let $\{U_i\}_{i \in I}$ be an open cover of A. Then by cptness, $A \subseteq \bigcup_{k=1}^{N} U_{i_k}$.

Then in particular, A is bdd from below by $\min\{U_{i_1}, \ldots, U_{i_N}\}$

Similarly for bdd. above. So A is bdd. and $\sup A$ exist

Remains to show $\sup A \in A$. Sps $\sup A \notin A$.

Then consider the open sets $U_n = (-\infty, \sup A - \frac{1}{n})$.

$\{U_n\}_{n\in\mathbb{N}}$ cover A: $\bigcup_{n\in\mathbb{N}} U_n = \bigcup_{n\in\mathbb{N}} (-\infty, \sup A - \frac{1}{n})$

$$= (-\infty, \sup A) \supseteq A.$$

Since A is cpt, there are $k_1, \dots, k_N$ s.t.

$$A \subseteq \bigcup_{l=1}^{N} (-\infty, \sup A - \frac{1}{k_l})$$

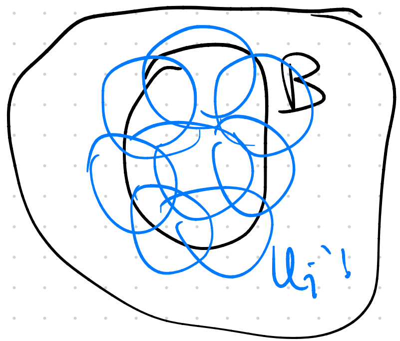
By nestedness, we have $< \sup A$

$$A \subseteq (-\infty, \sup A - \frac{1}{\max\{k_1, \dots, k_N\}}).$$

But this contradicts supremum of A

b) Closed and bounded: Since $B \subseteq A$, B is bdd. B is closed by assumption, so B is cpt.

Open cover: Let $\{U_i\}_{i \in I}$ be an open cover of B.



A

Then $\{U_i\} \cup B^c$ is an open cover of A

Note: B^c is open b/c. B is closed.

Then since A is cpt, A admits a finite subcover,

$$A \subseteq U_{i_1} \cup \dots \cup U_{i_N} \cup B^c$$

$$B \subseteq U_{i_1} \cup \dots \cup U_{i_N}$$. so B is cpt. //.

3. (Cantor's Intersection Theorem) Prove the following generalization of the Nested Interval Theorem for compact sets: Suppose $\{K_n\}_{n=1}^{\infty}$ is a sequence of nested non-empty compact subsets of $\mathbb{R}$. Then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots \supseteq K_n \supseteq K_{n+1} \supseteq \cdots$$

Pf: Closed and bounded: Since each K_n is bdd. and non-empty, $x_n := \sup K_n$ exist. Also, since $K_n \supseteq K_{n+1}$,

$$x_{n+1} \leq x_n.$$

Also x_n is contained in K_1, so $\{x_n\}$ is a bdd. seq.

So by MCT, $x := \lim_{n\to\infty} x_n$ exists.

WTS $x \in K_n$ for all n.

By 2.2a, $x_n \in K_n$ for each n. And by nested,

$$x_k \in K_n \text{ for each } k \geq n.$$

So for n fixed, the sequence $\{x_k : k \geq n\}$ is a convergent sequence entirely in K_n, so $x \in K_n$. ✓